Recursive formula for the peak delay time with travel distance in von Kármán type non-uniform random media on the basis of the Markov approximation

Tsutomu Takahashi,* Haruo Sato and Takeshi Nishimura

Department of Geophysics, Graduate School of Science, Tohoku University, Aramaki-aza Aoba 6-3, Aoba-ku, Sendai 980-8578, Japan. E-mail: ttaka@jamstec.go.jp

SUMMARY

Direct waves of microearthquakes in the high-frequency range (>1 Hz) strongly reflect the random inhomogeneities near their ray paths. This study conducts numerical simulations of envelope broadening of impulsively radiated wavelet assuming spatially non-uniform distribution of random inhomogeneities. We assume plural von Kármán type power spectral density functions (PSDF) for random inhomogeneity to clarify how the non-uniformly distributed random media affect the frequency dependence of envelope broadening. We employ the stochastic ray path method based on the Markov approximation for the mutual coherence function. This method is appropriate to simulate multiple forward scattering during the wave propagation. We mainly examine the travel distance and frequency dependence of the peak delay time in relation to the parameters characterizing the PSDFs. The peak delay time, which is defined as the time lag from the direct-wave onset to the maximum amplitude arrival of its envelope, is the best parameter reflecting the accumulated scattering effect in random media and is quite insensitive to the intrinsic attenuation. According to the numerical simulations in various non-uniform random media, we find some remarkable features in travel distance and frequency dependence, which cannot be found in uniform random media. For example, the frequency dependence in uniform random media is uniquely determined by the spectral gradient of PSDF for arbitrary travel distance; however, that in non-uniform media gradually changes as travel distance increases if the waves have experienced a change of spectral gradient in PSDF. Considering the results of our simulation, we propose a simple recursive formula to calculate the peak delay time in non-uniform random media. This recursive formula can predict the simulation results appropriately and relate the peak delay times to two parameters quantifying the von Kármán type PSDF in short wavelengths. It will become a mathematical base for the inversion of peak delay times of bandpass filtered traces to estimate the spatial distribution of random inhomogeneity spectra.

Key words: Wave scattering and diffraction.

1 INTRODUCTION

High-frequency seismic waves (>1 Hz) of microearthquakes usually show complex and incoherent wave trains as travel distance and/or lapse time increase owing to the medium inhomogeneities in the lithosphere. Due to such complexity, it is difficult to apply deterministic approaches to invert seismic waves for the earth structure. However, if we examine the envelopes of bandpass filtered seismograms, it becomes easier to reveal their common characteristics and their relations to the spectrum of random inhomogeneities near the unperturbed ray path from the stochastic point of view. For example, the duration time of direct wave monotonically increases with travel distance increasing due to the accumulation of scattering effect (e.g. Sato 1989; Atkinson & Boore 1995; Petukhin & Gusev 2003). The duration time usually depends on


534 © 2008 The Authors
Journal compilation © 2008 RAS
frequency and seismotectonic conditions near the ray path (e.g. Obara & Sato 1995; Takahashi et al. 2007). Obara & Sato (1995) analysed S-wave envelopes of microearthquakes in Kanto-Tokai, Japan. They reported that the S-wave envelopes observed in the backarc side of the volcanic front show longer duration times than those in the forearc side. They also pointed out that this characteristic becomes more significant at higher frequencies. Their study concluded that the medium inhomogeneities in short wavelengths are quite strong in the backarc side. Recently, Takahashi et al. (2007) precisely examined the path dependence of S-wave envelopes in northeastern Japan. According to their result, the S-wave envelopes for seismic rays propagating beneath the Quaternary volcanoes are more strongly broadened at higher frequencies. This characteristic implies that the medium inhomogeneity beneath Quaternary volcanoes is strong enough to broaden the impulsive wavelet.

The Markov approximation is a powerful stochastic method to describe the envelope of impulsive wavelet propagating through random media. This approach explicitly assumes that the velocity fluctuation is weak, the wavelength of incident wave is shorter than the correlation distance of random inhomogeneities, and the travel distance is much longer than the wavelength (e.g. Sato 1989). Studies using the Markov approximation have been successfully applied to describe the seismogram envelopes of microearthquakes observed in regional networks (e.g. Sato 1989; Saito et al. 2002). According to the studies based on the Markov approximation, the duration time of direct wave reflects the accumulated scattering effects (e.g. Sato 1989). Some quantitative relations have been derived to relate the stochastic parameters of the autocorrelation function (ACF) or the power spectral density function (PSDF) of random inhomogeneities to the parameters quantifying the duration time of the wave. These relations have been employed to estimate the lithospheric inhomogeneities from the observed envelopes of microearthquakes (e.g. Scherbaum & Sato 1991; Obara & Sato 1995; Saito et al. 2005).

The peak delay time is the most useful parameter quantifying the duration time of S-wave envelope (e.g. Gusev & Abubakirov 1999a,b; Takahashi et al. 2007). The peak delay time is defined as the time lag from the direct-wave onset to the maximum amplitude arrival of the envelope. This quantity is quite insensitive to the intrinsic attenuation, and never decreases during the wave propagation through random media. These characteristics allow us a simple speculation that a small peak delay time means the absence of strong inhomogeneities along the ray path (e.g. Takahashi et al. 2007). Gusev & Abubakirov (1999a, b) performed an inversion analysis of the peak delay time of observed data in Kamchatka area to investigate the layered structure of random inhomogeneities. They revealed that the crust has stronger inhomogeneities than the uppermost mantle. However, their method, which was originally proposed for the pulse broadening of optical waves by Bocharov (1985, 1987), is applicable to random media characterized by a Gaussian PSDF only.

According to the well-log data (e.g. Wu et al. 1994; Shiomi et al. 1997), we have to consider random inhomogeneities having power-law spectrum. Saito et al. (2002) have already extended the Markov approximation for the case of the von Kármán type PSDF which has a power-law spectrum at large wavenumbers. They pointed out the gradient of the PSDF strongly affects the frequency dependence of the envelope broadening. Recently, Saito et al. (2008) examined an extension of the Markov approximation for spatially non-uniform random media. They synthesized plane-wave envelopes in 2-D random media having Gaussian PSDFs by using two different approaches based on the Markov approximation. In the first approach, they numerically solved the stochastic differential equation for the two frequency mutual coherence function (TFMCF). In the second approach, they applied the stochastic ray path method (Williamson 1972; Sato & Korn 2007) considering the probability density function derived from the Markov approximation for mutual coherence function (MCF). The stochastic ray path method regards the intensity propagation in random media as successive ray bending processes. They confirmed that wave envelopes derived by both methods are consistent with those synthesized by the finite difference method. The stochastic ray path method can be easily extended for the wave propagation in 3-D random media having von Kármán type PSDF. This method is a powerful tool to synthesize wave envelopes in spatially non-uniform random media with short computational time.

In this study, we extend the stochastic ray path method developed in Saito et al. (2008) to impulsively radiated spherical wavelet propagation through 3-D random media having von Kármán type PSDF. Applying this method for the wave propagation through spatially non-uniform random media, which consist of different zones having different von Kármán type spectra, we propose a simple recursive formula to evaluate the peak delay time of envelope. This recursive formula is examined in the case of 3-D random media having Gaussian PSDFs by a comparison with that used in Gusev & Abubakirov (1999a).

2 STOCHASTIC RAY PATH METHOD FOR SPHERICAL WAVES IN RANDOM MEDIA

2.1 Mutual coherence function for spherical waves

The propagation of a scalar wave $u(x, t)$ in 3-D inhomogeneous media is governed by wave equation $[\Delta - \frac{\partial^2}{\partial t^2} V(x)^2] u(x, t) = 0$ where $\Delta$ is the Laplacian. The velocity $V(x)$ is written as $V(x) = V_0 \{1 + \xi(x)\}$ where $V_0$ is the average velocity and $\xi(x)$ is the fractional fluctuation. For wave radiation from a point source located at the origin, wave $u(x, t)$ is represented by a superposition of harmonic spherical waves with amplitude $U(r, \theta, \phi, \omega)$ at angular frequency $\omega$ in the spherical coordinate system (e.g. Saito et al. 2002). We measure angle $\theta$ from the unperturbed ray path. If the velocity fractional fluctuation $\xi(x)$ is small and the wavelength is shorter than the correlation distance $a$ of inhomogeneity, the wave equation is reduced to the parabolic wave equation for $U(r, \theta, \phi, \omega)$ near the unperturbed ray path as

$$2i k \frac{\partial}{\partial r} U(r, \theta, \phi, \omega) + \frac{1}{r^2} \Delta_\perp U(r, \theta, \phi, \omega) - 2k_0 \xi(r, \theta, \phi) U(r, \theta, \phi, \omega) = 0,$$

where wavenumber $k_0 = \omega / V_0$ and $\Delta_\perp \approx \frac{1}{r^2} (\theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}$ is the transverse Laplacian for $\theta < 1$ (Saito et al. 2002).
We consider an ensemble average of random media, where the fractional fluctuation $\xi(x)$ is a random function of space, and satisfies the condition that its ensemble average $\langle \xi(x) \rangle$ is zero. The randomness is statistically characterized by the autocorrelation function (ACF) of fractional fluctuation $\xi(x)$ as $R(x) \equiv \langle \xi(y) \xi(y + x) \rangle$. We assume the randomness is homogeneous and isotropic. Then, we define the mutual coherence function (MCF) of wavefield $U(r, \theta, \phi, \omega)$ at distance $r$ and at angular frequency $\omega$ as $\Gamma_1(\mathbf{r}_{\perp}, \mathbf{r}_{\perp}, r, k_0) = \langle U(\mathbf{r}_{\perp}, r, \omega) U^*(\mathbf{r}_{\perp}, r, \omega) \rangle$ (e.g. Sato & Fehler 1998), where the asterisk means complex conjugate, and $\mathbf{r}_{\perp}$ represents the location on the transverse plane that is normal to the unperturbed ray path (see fig. 2 in Saito et al. 2002). The average intensity of the wave is written by using the MCF as

$$
\tilde{I}_0(r, k_0) = \langle |U(r, \theta, \phi, \omega)|^2 \rangle = \frac{1}{(2\pi)^2 r^2} \Gamma_1(\mathbf{r}_{\perp}, \mathbf{r}_{\perp}, r, k_0),
$$

where $\mathbf{r}_{\perp}$ is a lag distance on the transverse plane at distance $r$. We may interpret the time trace of the intensity as the mean square (MS) envelope at a given angular frequency. By taking ensemble average of the parabolic wave equation (1) and neglecting backward scattering (Sato & Fehler 1998, p. 244), we can obtain a differential equation for the MCF as

$$
\frac{\partial}{\partial r} \Gamma_1 + k_0^2 [ A(0) - A(\mathbf{r}_{\perp}) ] \Gamma_1 = 0,
$$

where $A(r)$ is the longitudinal integral of the ACF as $A(\mathbf{r}_{\perp}) = \int_{-\infty}^{\infty} R(\mathbf{r}_{\perp}, z) \, dz$.

Following the stochastic ray path method in 2-D random media (Sato & Korn 2007; Saito et al. 2008), we derive the equation which relates the wave scattering in a thin spherical layer to the development of the MCF. Integrating eq. (3), we obtain the MCF for increment $\Delta r$ as

$$
\Gamma_1(\mathbf{r}_{\perp}, r + \Delta r, k_0) = \Phi(\mathbf{r}_{\perp}, \Delta r, k_0) \Gamma_1(\mathbf{r}_{\perp}, r, k_0),
$$

where

$$
\Phi(\mathbf{r}_{\perp}, \Delta r, k_0) = e^{-k_0^2 [ A(0) - A(\mathbf{r}_{\perp}) ] \Delta r}.
$$

This term can be regarded as the transfer function of the MCF. Applying the Fourier transform on the transverse plane, we obtain

$$
\tilde{\Gamma}_1(\mathbf{k}_{\perp}, r + \Delta r, k_0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk \tilde{\Phi}(\mathbf{k}_{\perp} \rightarrow k'_{\perp} - k_{\perp}, \Delta r, k_0) \tilde{\Gamma}_1(\mathbf{k}_{\perp}, r, k_0),
$$

where

$$
\Gamma_1(\mathbf{r}_{\perp}, r, k_0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk \tilde{\Phi}(\mathbf{k}_{\perp}) \tilde{\Gamma}_1(\mathbf{k}_{\perp}, r, k_0).
$$

We introduce a new parameter $\mathbf{s} = (s_x, s_y)$ representing ray-direction measured from the unperturbed ray-direction as $s_x \equiv k_x/k_0 = \sin \theta_x$ and $s_y \equiv k_y/k_0 = \sin \theta_y$, where $\theta_x$ and $\theta_y$ are scattering angle projected on the $x-z$ and $y-z$ planes of the local Cartesian coordinate system, respectively (Fig. 1). By using this parameter $\mathbf{s}$, we redefine functions $\tilde{\Gamma}_1(\mathbf{s}, r, k_0)$ and $\tilde{\Phi}_s(\mathbf{s}, r, k_0)$ as

$$
\tilde{\Gamma}_1(\mathbf{s}, r, k_0) = \frac{k_0^2}{(2\pi)^2} \tilde{\Gamma}_1(0,0, r, k_0),
$$

and

$$
\tilde{\Phi}_s(\mathbf{s}, r, k_0) = \frac{k_0^2}{(2\pi)^2} \tilde{\Phi}(0,0, r, k_0).
$$

Then, we can rewrite eq. (6) as

$$
\tilde{\Gamma}_1(\mathbf{s}, r + \Delta r, k_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ds' \tilde{\Phi}_s(\mathbf{s} - \mathbf{s}', \Delta r, k_0) \tilde{\Gamma}_1(\mathbf{s}', r, k_0).
$$

We finally obtain the following relation

$$
\tilde{I}_0(r, k_0) = \frac{1}{r} \int_{-1}^{1} \int_{-1}^{1} ds' \tilde{\Gamma}_1(\mathbf{s}, r, k_0).
$$

The $\tilde{\Gamma}_1(\mathbf{s}, r, k_0)$ in eq. (10) is the angular spectrum, and the $\tilde{\Phi}_s$ can be interpreted as a probability density function of ray scattering angle $\mathbf{s}$ in a thin layer. Eq. (11) means that the integral of $\tilde{\Gamma}_1(\mathbf{s}, r, k_0)$ over $\mathbf{s}$ gives us the intensity of wavefield. In the following, we describe how we use these two equations to derive MS envelopes for an impulsive radiation from a point source.

### 2.2 Numerical simulation of MS envelopes based on the stochastic ray path method

We use a stochastic ray path method that numerically simulates the propagation of intensity particles by means of the Monte Carlo method, tracking the traveltime of each particle (Williamson 1972). To apply this method for spherical waves, we consider many thin spherical layers. The centre of these spherical layers corresponds to a seismic source. In our simulation, intensity particles are radiated from the source with a fixed radiation angle as shown in Fig. 2. These particles are randomly scattered at layer boundaries, and finally reach to a layer boundary at distance $r$. 

© 2008 The Authors, GJI, 173, 534–545

Journal compilation © 2008 RAS
The travel distance of each intensity particle depends on accumulated scattering angles. In the nth spherical layer, a particle which arrived at the boundary on \( r_{n-1} \) with an incident angle \( \phi'_{n-1} \) is randomly scattered, and changes its propagation direction as \( \phi_n = \phi'_{n-1} + \text{random} \). This random angle is statistically determined by giving the random numbers for \( s \) by the probability density function \( \Phi_s \). The travel distance \( \Delta l_n \) in the nth spherical layer (Fig. 1) can be written as

\[
\Delta l_n = \sqrt{r_n^2 + r_{n-1}^2 - 2r_n r_{n-1} \cos (\phi_n - \phi_n')} \approx \sqrt{r_n^2 - 2r_n r_{n-1} + r_{n-1}^2 + r_{n-1} \left( \phi_n - \phi_n' \right)^2} \frac{2\sqrt{r_n^2 - 2r_n r_{n-1} + r_{n-1}^2}}{2}.
\]

In the nth layer, angles \( \theta_s \) and \( \theta_s' \) are the projections of \( \phi_n \) on the \( x-z \) and \( y-z \) planes, respectively. Similarly, angles \( \theta_x' \) and \( \theta_y' \) are the projections of \( \phi_n' \). These angles satisfy the following relation:

\[
cos (\phi_n - \phi_n') = \frac{\cos^2(\theta_s - \theta_s') + \cos^2(\theta_x - \theta_x') - 1}{\cos^2(\theta_s - \theta_s' / r_n) + \cos^2(\theta_x - \theta_x' / r_n)} - 1,
\]

where we use the small angle approximation for \( \theta_s \) and \( \theta_x' \): \( \theta' = \arcsin(\sin(\theta_{n-1} / r_n)) \approx \theta_{n-1} / r_n \). In our simulation, we evaluate the scattering angles \( \theta_s \) and \( \theta_x' \) independently in each thin layer by using the probability density function \( \Phi_s \), then calculate \( \Delta l_n \) by using eqs (12) and (13). The traveltime of each particle is given by the accumulated path lengths divided by the background velocity \( V_0 \). We interpret the histogram of traveltimes for all intensity particles at the receiver as the time trace of wave intensity, which corresponds to the MS envelope for a given source radiation. The MS envelope derived by this method is without wandering effect (Sato & Fehler 1998). We note that the traveltime histogram of all the intensity particles reached on the spherical surface of radius \( r \) for a fixed source radiation gives the MS envelope at a station for the case of isotropic source radiation.

In the following sections, we show some examples of envelopes in spatially uniform and non-uniform random media synthesized by the stochastic ray path method. The number of particles used in simulation is 1000000, and we take the thickness of a layer \( \Delta r = 2 \) km and the average velocity \( V_0 = 4.0 \) km \( s^{-1} \). We use the Mersenne Twister (Matsumoto & Nishimura 1998) as a pseudorandom number generator to secure the uniform distribution of random numbers in the Monte Carlo simulation.

The von Kármán type ACF is used for random inhomogeneity in the following simulation:

\[
R (x) = R (x) = \frac{x^{2+1-k}}{\Gamma (k)} \left( \frac{x}{a} \right)^k \left( \frac{x}{a} \right)^k.
\]

where the corresponding PSDF is

\[
P (m) = \frac{8\pi^{3/2} a^2 k^2 \Gamma (k+3/2)}{\Gamma (k) (1+a^2 m^2)^{3/2}},
\]

where \( \varepsilon \) is the root mean square (rms) of the velocity fractional fluctuation, \( \Gamma \) is the gamma function, \( K \) is the modified Bessel function, and the parameter \( k \) controls the roll-off of PSDF in short wavelength (Sato & Fehler 1998; Saito et al. 2002). Then, the longitudinal integral of ACF is given by

\[
A (r_{ld}) = \frac{2^{-k+3/2} \sqrt{\pi} \varepsilon^2 a}{\Gamma (k)} \left( \frac{r_{ld}}{a} \right)^{k+1/2} K_{k+1/2} \left( \frac{r_{ld}}{a} \right).
\]
Figure 2. Schematic illustration of envelope synthesis by the stochastic ray path method. The right-hand column shows the cartoon of envelope broadening as travel distance increases. The left-hand column represents the propagation process of particles. The radiation angle at the source is fixed as a certain direction. On the sphere of radius $r_n$, many particles arrive at various points. The histogram of traveltime for all particles corresponds to MS envelope at the station on the distance $r$.

The term $A(0) - A(r_{\perp d})$, which appears in eq. (5), is calculated following the approximation representation for short transverse distances $r_{\perp d}/a$ from $10^{-4}$ to $10^{-1}$ [see eqs (18), (19) and (20) in Saito et al. 2002].

$$A(0) - A(r_{\perp d}) \approx \varepsilon^2 a C(\kappa) \left( \frac{r_{\perp d}}{a} \right)^{p(\kappa)}, \quad \text{for} \quad \frac{r_{\perp d}}{a} \ll 1.$$  \hspace{1cm} (17)

The values of $p(\kappa)$ and $C(\kappa)$ are listed in Table 1. The probability density $\Phi_s$ is numerically calculated by applying the 2D-FFT for eq. (5). Fig. 3 shows the plot of $\Phi_s$ for $\kappa = 0.3$, $\varepsilon = 0.025$, $a = 5$ km and $k_0 = 16$ km$^{-1}$. We can recognize the $\Phi_s$ rapidly decreases as $\sqrt{s_x^2 + s_y^2}$ increases.

3 ENVELOPE BROADENING IN UNIFORM RANDOM MEDIA

3.1 Envelopes in uniform random media

Fig. 4 shows the rms envelopes synthesized by the stochastic ray path method for a point source radiation in spatially uniform random media characterized by von Kármán type ACF. We assume the predominant frequency of incident wave is 10 Hz. The time trace of the square root of wave intensity corresponds to the rms envelope. The correlation distance $a$ of inhomogeneities is chosen to be 5 km, the values of $\kappa$ are ranging from 0.2 to 0.9 with a step of 0.1, and $\varepsilon$ are 0.02, 0.04 and 0.06. The grey lines represent rms envelopes synthesized by the stochastic ray path method. Black thin lines are the envelopes calculated by using the Markov approximation for the two-frequency mutual coherence.
Table 1. Parameters $b_0(\kappa)$, $C(\kappa)$ and $p(\kappa)$ in eqs (17) and (20).

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$b_0(\kappa)$</th>
<th>$C(\kappa)$</th>
<th>$p(\kappa)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.10</td>
<td>0.56</td>
<td>1.19</td>
</tr>
<tr>
<td>0.20</td>
<td>0.17</td>
<td>1.06</td>
<td>1.38</td>
</tr>
<tr>
<td>0.30</td>
<td>0.23</td>
<td>1.56</td>
<td>1.56</td>
</tr>
<tr>
<td>0.40</td>
<td>0.28</td>
<td>2.00</td>
<td>1.71</td>
</tr>
<tr>
<td>0.50</td>
<td>0.31</td>
<td>2.28</td>
<td>1.83</td>
</tr>
<tr>
<td>0.60</td>
<td>0.34</td>
<td>2.31</td>
<td>1.91</td>
</tr>
<tr>
<td>0.70</td>
<td>0.36</td>
<td>2.14</td>
<td>1.95</td>
</tr>
<tr>
<td>0.80</td>
<td>0.37</td>
<td>1.90</td>
<td>1.98</td>
</tr>
<tr>
<td>0.90</td>
<td>0.37</td>
<td>1.68</td>
<td>1.99</td>
</tr>
<tr>
<td>1.00</td>
<td>0.38</td>
<td>1.50</td>
<td>1.99</td>
</tr>
</tbody>
</table>

Note: These values are for the transverse distance $r_{\perp}/a$ from $10^{-4}$ to $10^{-1}$ (Saito et al. 2002). $C(\kappa)$ and $p(\kappa)$ are referred from Saito et al. (2002) and $b_0(\kappa)$ is defined in this study, by considering the envelopes derived by the stochastic differential equation for TFMCF.

Figure 3. Plot of $\Phi_s$ for $\kappa = 0.3$, $\varepsilon = 0.025$, $a = 5$ km and $k_0 = 16$ km$^{-1}$. The $k_0 = 16$ km$^{-1}$ corresponds to the wavenumber of 10 Hz for $V_0 = 4.0$ km s$^{-1}$.

Figure 4. The rms envelopes at travel distance 200 km in random medium characterized by von Kármán type PSDF ($a = 5$ km) and the constant background velocity $V_0 = 4.0$ km s$^{-1}$ for different $\varepsilon$ values (a) 0.02, (b) 0.04 and (c) 0.06. The value of $\kappa$ ranges from 0.2 to 0.9 with a step 0.1. Grey curves are simulation results by our stochastic ray path method. Black ones are derived by using the Markov approximation for the TFMCF (Saito et al. 2002).

function (TFMCF) (Saito et al. 2002). These envelopes, which are derived by two different methods, show good agreement. However, closely looking at the envelopes, we find that the envelopes for strongly inhomogeneous media having small $\kappa$ and large $\varepsilon$, that is, having rich spectra at short wavelengths show some discrepancies between the two methods. This is because large angle scattering, which cannot satisfy the small angle approximation in eq. (13), becomes dominant. In the following, therefore, we apply our simulation only for the case of $\kappa \geq 0.3$.

3.2 Peak delay times in uniform random media

The peak delay time in uniform random media can be represented by a simple form derived from the Markov approximation for TFMCF (Saito et al. 2002). The peak delay time for an impulsive source radiation in a spatially uniform von Kármán random medium can be written as a function of travel distance $r$ and frequency $f$

$$t_p = N_p(\kappa, \varepsilon, a) \cdot f^{2M_p(\varepsilon) - 4} \cdot r^{M_p(\varepsilon)}.$$  

(18)
Parameters $M_p(\kappa)$ and $N_p(\kappa, \varepsilon, a)$ are related to the stochastic parameters of the von Kármán type PSDF as follows:

$$M_p(\kappa) = 1 + \frac{2}{p(\kappa)},$$

and

$$N_p(\kappa, \varepsilon, a) = b_p(\kappa) \left\{ \frac{C(\kappa)}{2} \int_0^{1-\epsilon/p(\kappa)} \left[ \frac{2}{\pi a} \frac{(2\pi)^{-1+1/p(\kappa)}}{a} \right]^2 \right\}.$$

This representation is derived from the relation $t_p = b_p(\kappa) t_M$, where $t_M$ is the characteristic time (Saito et al. 2002)

$$t_M = \frac{C(\kappa)}{2} \frac{1}{\int_0^{1-\epsilon/p(\kappa)} \left[ \frac{2}{\pi a} \frac{(2\pi)^{-1+1/p(\kappa)}}{a} \right]^2 \right\}.$$

The values of $b_p(\kappa)$ are listed in Table 1. The parameters $p(\kappa)$ and $b_p(\kappa)$ monotonically increase as $\kappa$ increases, and $C(\kappa)$ takes the maximum at $\kappa = 0.6$. Eq. (18) implies that the travel distance dependence and the frequency dependence of the peak delay time are controlled by only $\kappa$. The parameter $M_p$, which directly corresponds to the power of travel distance dependence, monotonically increases from 2.0 to 2.68 as $\kappa$ decreases from 1.0 to 0.1. Eq. (18) faithfully describes the peak delays of synthesized envelopes in uniform random media.

### 4 ENVELOPE BROADENING IN NON-UNIFORM RANDOM MEDIA

#### 4.1 Envelopes in non-uniform random media

We apply the stochastic ray path method for a random medium which consists of three types of random inhomogeneities having different von Kármán type PSDFs. The boundaries of different inhomogeneities are located at distances 120 and 200 km from a point source (see Fig. 5b). For brevity, these three parts are named as zone-1 ($r = 0–120$ km), zone-2 ($r = 120–200$ km) and zone-3 ($r = 200–300$ km) from the source to the receiver. The randomness in zone-1 is characterized by $\kappa = 0.8$ and $\varepsilon = 0.04$, that in zone-2 is $\kappa = 0.5$ and $\varepsilon = 0.06$, and that in zone-3 is $\kappa = 0.7$ and $\varepsilon = 0.05$. All inhomogeneities have the same correlation distance $a = 5$ km and the background velocity $V_0 = 4.0$ km s$^{-1}$. The PSDFs of three zones are plotted in the inserted graph in Fig. 5(a), where the vertical arrow represents the wavenumber ($16$ km$^{-1}$) corresponding to the predominant frequency 10 Hz. The power spectrum of random inhomogeneity is richer at short wavelengths in zone-2 than the other zones. In Fig. 5(a), black lines represent rms envelopes in this non-uniform random media. For comparison, grey lines

![Figure 5](image-url)
Envelope broadening in non-uniform random media

4.2 Peak delay time in non-uniform random media

We measure the peak delay times of envelopes for every 10 km in the non-uniform random medium in Fig. 5. Fig. 6 shows the logarithmic plot of the peak delay time against the travel distance. Open circle is the average of 10 trials of simulation at each distance. We can recognize a clear bend of linear trend at each boundary between different inhomogeneities. The exponent of the travel distance dependence in zone-2 (120–200 km), in which \( \kappa = 0.5 \), is approximately 3.4, while the \( M_p \) for \( \kappa = 0.5 \) is approximately 2.09 as shown in Fig. 6 with the dashed line. This significant change of the exponent is caused by the change of random inhomogeneities.

5 RECURSIVE FORMULA OF THE PEAK DELAY TIME IN NON-UNIFORM RANDOM MEDIA

Eq. (18) is applicable for an impulsive source radiation in uniform random media. If the PSDF of random inhomogeneities changes along the ray path, we must take account of the incidence of broadened waves to each zone having different inhomogeneity. Here, we propose a simple representation of the peak delay time in non-uniform random media as follows.

We first study the simplest example; there are two different zones between source and receiver as shown in Fig. 7. The medium between the seismic source and distance \( r_1 \) (zone-1) has weak inhomogeneities, and that between distances \( r_1 \) and \( r_2 \) (zone-2) has strong inhomogeneities. Assuming an impulsive source radiation, we can calculate the peak delay time \( t_p \) at distance \( r_1 \) by applying eq. (18) as

\[
t_p^{(1)} = N_p^{(1)} f_2 M_p^{(1)} f_1^{4 M_p^{(1)}}.
\]

To obtain the peak delay time \( t_p \) at distance \( r_2 \), we replace the inhomogeneity in zone-1 with that in zone-2 according to the following procedure. Since the peak delay time on the boundary between zone-1 and 2 must be the same as that between the original and replaced media, we can calculate an equivalent travel distance \( r_1' \) in the replaced medium as,

\[
r_1' = \left[ \frac{t_p^{(1)}}{N_p^{(2)} f_2^{2 M_p^{(2)} f_1^{4 M_p^{(2)}}}} \right]^{1/M_p^{(2)}}.
\]
where \( r'_1 \) is defined as the distance at which the peak delay time becomes \( t^{(1)}_p \) in the medium characterized by PSDF in zone-2 for an impulsive source radiation. By using this equivalent travel distance \( r'_1 \), we may calculate the peak delay time at the receiver in zone-2 as

\[
t^{(2)}_p = N^{(2)}_p f_{2}^{2MP^{(2)}_p - 4} \left[ r'_1 + (r_2 - r_1) \right]^{M^{(2)}_p}.
\]  

(24)

For the case that the random medium consists of many zones having different PSDF, it is enough to repeat the similar replacement from source to receiver sequentially. Finally, we get the following recursive formula for different \( n \) and \( (n-1) \) th zones as,

\[
t^{(n)}_p = N^{(n)}_p f_{2}^{2MP^{(n)}_p - 4} \left[ r'_{n-1} + (r_n - r_{n-1}) \right]^{M^{(n)}_p},
\]  

(25)

\[
r'_{n-1} = \left[ \frac{t^{(n-1)}_p}{N^{(n-1)}_p f_{2}^{2MP^{(n-1)}_p - 4}} \right]^{1/M^{(n)}_p}.
\]  

(26)

The thick line in Fig. 6 is the theoretical prediction by the recursive formula: eqs (22), (25) and (26). This example shows that our recursive formula appropriately predicts the peak delay time calculated from numerical simulations.

We have assumed the same correlation distance \( a \) for different zones in the simulation shown in Figs 5 and 6. We note that the simulation results assuming different correlation distance \( a \) can also be predicted by the above mentioned recursive formula. This is because the ratio \( e^2/(\kappa \nu^{p-1})a^{-1} \) in \( N_p \) is a fundamental parameter describing the multiple forward scattering in random media characterized by von Kármán type PSDF. Fig. 8 shows plots of PSDFs for different \( a \) and \( \kappa \) values for the case of \( e^2/(\kappa \nu^{p-1})a^{-1} = 10^{-4} \) km\(^{-1}\). The difference of the correlation distance \( a \) generates significant variation of the PSDFs in small wavenumbers. However, spectral amplitudes at large wavenumbers higher than the corner are almost the same each other especially for small \( \kappa \) (Fig. 8a), although we may find slight difference of PSDFs at large wavenumbers for the case of large \( \kappa \) (Fig. 8b). This coincidence implies that the \( e^2/(\kappa \nu^{p-1})a^{-1} \) is a parameter characterizing the power spectrum at large wavenumbers (i.e. short wavelengths). Therefore, we can safely discuss the significant difference of PSDF in short wavelengths from the values of \( \kappa \) and \( e^2/(\kappa \nu^{p-1})a^{-1} \), even though we do not have enough information on the correlation distance \( a \). Consequently, we can say the parameter set \((M_p, N_p)\) or \((\kappa, e^2/(\kappa \nu^{p-1})a^{-1})\) can be used for the calculation of the peak delay time in non-uniform random media, and the estimation of the PSDF in short wavelengths (at large wavenumbers) with little information on \( a \).

We note that the replacement of the travel distance from \( r_{n-1} \) to \( r'_{n-1} \) in eq. (26) deforms the wave front shape of spherical wave due to the change of spherical radius. The good coincidence of our method with the simulation results implies that this effect is negligibly weak as long as the variation of inhomogeneities is not so large. This is because the peak delay can be affected by inhomogeneities only in the vicinity of the unperturbed ray path. The possible breakdown of our recursive formula is examined by applying our formula and the stochastic ray path method for about 70 different non-uniform random media. The statistical parameters of non-uniform random media are in the range of \( \kappa = 0.3-0.9 \) and \( \varepsilon = 0.005-0.10 \), where we fix \( a = 5 \) km like previous simulations. Each random medium contains from two to four zones of different random medium of which the boundaries are located at 80–200 km from the source and the zone thickness is larger than 40 km. Since \( t_p \) values measured from simulations by using the stochastic ray path method show fluctuations, we examined the possible breakdown.
under the condition that our method is in the range of scatter of \( t_p \) for 10 trials of numerical simulation. These examinations don’t indicate any clear relation between the difference of individual parameters (\( \kappa \) or \( \varepsilon \)) at boundaries and whether the breakdown happened or not. However, we find that the breakdown of our method happened in the case that the ratio of the original travel distance \( r_i \) to the equivalent one \( r_i' \) or the reciprocal \( r_i'/r_i \), is larger than 3.0. If \( r_i' \) is much smaller than \( r_i \), our method overestimates the peak delay time compared to the numerical simulation results. In the opposite case, we can expect the underestimation of the peak delay time.

### 6 DISCUSSION

We have demonstrated that our recursive formula successfully predicts the peak delay time in spatially non-uniform random media characterized by von Kármán type PSDFs. In the case of Gaussian PSDF, the method employed in Gusev & Abubakirov (1999a, b), which is originally proposed by Bocharov (1985), is available to calculate the peak delay time. This approach seems to be straightforward without introducing the equivalent travel distance in non-uniform media, even though its application is limited for the Gaussian PSDF. Here, we compare our recursive formula with the methods of Gusev & Abubakirov (1999a). Then, we describe the frequency dependence of the peak delay time in non-uniform random media having von Kármán type PSDFs, and its implication for the inversion analysis of the peak delay times.

#### 6.1 Comparison with the method of Gusev & Abubakirov (1999a)

Gusev & Abubakirov (1999a) uses a simple integral form relating the medium inhomogeneities to the mean delay time of seismic energy. Considering that the ratio of the mean delay time and the peak delay time is constant in uniform random media having Gaussian PSDF (Gusev & Abubakirov 1996), we can rewrite their integral form as

\[
t_p = \frac{1}{2r} \int_0^\infty g'_e(r - \eta) \, d\eta
data distance \( r \). Their \( g'_e \) can be written as
\[
g'_e \approx 2.20 \sqrt{\pi \varepsilon^2 / (a V_0)}
\]

according to the relation

\[
t_p \approx 0.367 \sqrt{\pi \varepsilon^2 \rho / 2a V_0}
\]

derived from the Markov approximation for spherical waves in 3-D random media having Gaussian PSDF (e.g. Shishov 1974; Saito et al. 2002). We numerically synthesized the envelopes in the random media having the following three different Gaussian PSDFs: PSDF characterized by \( \varepsilon = 0.04 \) for zone-1 (\( r = 0–120 \) km), PSDF by \( \varepsilon = 0.06 \) for zone-2 (\( r = 120–200 \) km) and PSDF by \( \varepsilon = 0.04 \) for zone-3 (\( r = 200–300 \) km). The correlation distance \( a = 5 \) km and the average velocity \( V_0 = 4.0 \) km s\(^{-1}\) are the same for three zones. Open circles in Fig. 9 are the peak delay time measured from the envelopes synthesized by the stochastic ray path method. The dashed line is the prediction by Bocharov’s method employed in Gusev & Abubakirov (1999a), and the thick black line is the prediction by our recursive formula assuming \( M_p = 2.0 \) and \( N_p = 0.367 \sqrt{\pi \varepsilon^2 / 2a V_0} \). These two coefficients are derived from the comparison with eqs (18) and (27). We note that \( \kappa = 1.0 \) for von Kármán type PSDF also indicates \( M_p = 2.0 \), but the \( N_p \) is different between Gaussian and von Kármán type PSDFs. Both of the methods appropriately predict the simulated result, and the difference of two methods seems to be minor. The slight discrepancy of simulation result and the prediction by Bocharov’s method in zone-3 may imply that the assumption of constant ratio of the mean delay to the peak delay is not valid in non-uniform random media. Considering the good coincidence of the Markov approximation with the numerical simulation results (e.g. Fehler et al. 2000; Saito et al. 2003, 2008), we can say that our method can be used as an approach to predict the peak delay times. Consequently, the recursive formula in this study appropriately predicts the peak delay time in random media having Gaussian PSDF as like as Gusev & Abubakirov (1999a). An advantage of our method is the broader applicability to von Kármán type PSDF of random media.

---

**Figure 8.** Change of PSDFs for the constant \( \varepsilon^{2/1(\kappa-1)} a^{-1} \) and different values of \( a \): 1, 3 and 5 km for (a) \( \kappa = 0.3 \) and (b) \( \kappa = 0.7 \). The \( \varepsilon^{2/1(\kappa-1)} a^{-1} \) is \( 10^{-4} \text{ km}^{-1} \) and the background velocity is \( V_0 = 4.0 \text{ km s}^{-1} \). The PSDF is calculated from eq. (15).
Figure 9. Peak delay time in a spatially non-uniform random media characterized by Gaussian PSDF. Open circle is the average of 10 trials of numerical simulation. Black line is the theoretical values calculated by using the recursive formula proposed in this study, and the dashed line is the prediction by the method of Gusev & Abubakirov (1999a).

6.2 Frequency dependence of the peak delay time in non-uniform random media

As shown in eq. (18), the peak delay time in spatially uniform random media increases with the power of travel distance and frequency. These powers are functions of $\kappa$. However, in non-uniform random media, the power of travel distance dependence is affected by all the parameters characterizing random inhomogeneities through which seismic waves have propagated, as we mentioned in previous sections. Here, we examine the frequency dependence in non-uniform random media from the perspective to the relation to the $\kappa$.

Open circles in Fig. 10 show the numerical simulation results of the peak delay times in non-uniform random media having the von Kármán type PSDF for three different frequencies: 5, 10 and 20 Hz. The non-uniform random medium in Fig. 10(a) contains three inhomogeneity zones characterized by different $\kappa$ values. The values of $\kappa$ are 0.4, 0.8 and 0.3 in zone-1, 2 and 3, respectively. In zone-1 (0–120 km), the ratios of the peak delay time between the different frequencies are constant. In zone-2, the ratios become small as travel distance increases. In zone-3, the ratios gradually increase as travel distance increases due to the decrease of $\kappa$. Fig. 10(b) is for the case that $\kappa$ is constant ($\kappa = 0.4$). The values of $\varepsilon$ are 0.5, 0.35 and 0.5 in zone-1, 2 and 3, respectively. Fig. 10(b) clearly shows the ratios of the peak delay time between different
frequencies are kept constant regardless the change of $\epsilon$. These simulation results indicate that the frequency dependence in non-uniform random media is also controlled by only $\kappa$. It implies that the frequency dependence of the peak delay times make it possible to estimate the parameter $\kappa$. Frequency dependence of $t_p$ has been reported in northeastern Japan (Obara & Sato 1995; Takahashi et al. 2007) as station averaged values. Such averaged characteristics of the peak delay time are useful for the investigation of the lithospheric inhomogeneities.

7 CONCLUSIONS

This study examined the peak delay time of seismic wave propagating through spatially non-uniform random media characterized by von Kármán type PSDFs. To simulate the envelope broadening of high-frequency waves for long travel distances, we extended the stochastic ray path method for the spherical waves based on the Markov approximation for the mutual coherence function of wavefield. From the simulation results in various non-uniform random media, we revealed that the relation between the parameter $\kappa$ and the peak delay time is quite different between uniform and non-uniform random media. In uniform media, the power of the travel distance dependence is a function of $\kappa$. On the other hand, the travel distance dependence in non-uniform random media is affected by the stochastic parameters of random inhomogeneities in which waves propagated. To describe such simulation results, we newly proposed a simple recursive formula which enables us to predict the peak delay time for non-uniform random media. This recursive formula can predict the peak delay time regardless the number of different inhomogeneities along the ray path as long as the spatial variation of inhomogeneities is smooth. We further show that the frequency dependence of the peak delay time directly reflects the value of $\kappa$ in non-uniform random media. This suggests that inversion analysis of the peak delay times of $S$-wave envelopes of microearthquakes for the earth medium inhomogeneity can be possible by using our recursive formula considering the frequency dependence of the peak delay times.

ACKNOWLEDGMENTS

The authors are grateful to Michael Korn, Michael Fehler and Alexander A. Gusev for their helpful comments. This work is partially supported by the grant in aid for Scientific Research of JSPS #15540399, and the sponsorship of JNES open application project for enhancing the basis of nuclear safety, and the ERI cooperative research program. This work was conducted as a part of the 21COE program on Earth Science of Tohoku University. GMT code is used for figure plotting.

REFERENCES


